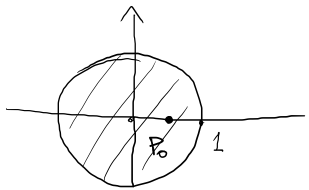
Kruhový talíř o rovnici $x^2 + y^2 \leq 1$ je zahřátý na teplotu $T(x, y) = x^2 + 2y^2 - x$. Najděte nejteplejší a nejstudenější bod na talíři.



$D = \{(x,y) : x^2+y^2 \leq 1\}$

$\nabla T = \langle 2x-1, 4y \rangle$ $\quad \nabla T = \vec{o}$ $\quad P_0\left(\frac{1}{2}, 0\right) \in D^\circ$

$\partial D = \{(x,y) : x^2+y^2 = 1\} = \{(x,y) : g(x,y) = 0,\ g(x,y) = x^2+y^2-1\}$

$L(x,y,\lambda) = T(x,y) + \lambda g(x,y) = x^2+2y^2-x+\lambda(x^2+y^2-1)$

$\nabla L = \vec{o}$

$\frac{\partial L}{\partial \lambda} = 0$

$$\begin{cases} 2x-1+2\lambda x = 0 \\ 4y+2\lambda y = 0 \\ x^2+y^2 = 1 \end{cases} \quad \begin{cases} 2x(1+\lambda) = 1 \\ 2y(2+\lambda) = 0 \\ x^2+y^2 = 1 \end{cases}$$

$\begin{cases} y = 0 \\ x^2+y^2 = 1 \end{cases}$ $\quad x^2 = 1 \quad x = \pm 1 \quad P_1(1,0) \quad P_2(-1,0)$

$\begin{cases} \lambda = -2 \\ x = -1/2 \\ y^2 = 1 - \frac{1}{4} = \frac{3}{4},\ y = \pm\frac{\sqrt{3}}{2} \end{cases}$ $\quad P_3\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right),\ P_4\left(-\frac{1}{2}, +\frac{\sqrt{3}}{2}\right)$

$T(P_0) = -\frac{1}{4}$

$T(P_1) = 0$

$T(P_2) = 2$

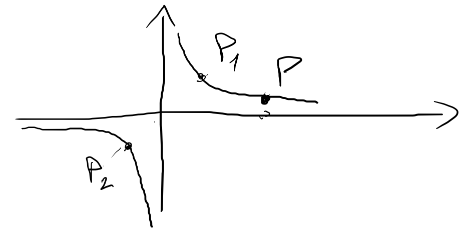
$T(P_3) = \frac{1}{4} + \frac{3}{2} + \frac{1}{2} = \frac{9}{4}$

$T(P_4) = \frac{1}{4} + \frac{3}{2} + \frac{1}{2} = 9/4$

P_3 a P_4 jsou body maxima

P_0 je bod minimum.

Vyšetřete extrémy funkce f na zadané množině M.



$$f = x^2 + y^2,\ M = \{(x,y) \in \mathbb{R}^2 \mid xy = 1\},$$

$M = \{(x,y) : g(x,y) = 0 \quad g(x,y) = xy - 1\}$

$L(x,y,\lambda) = f + \lambda g = x^2 + y^2 + \lambda(xy - 1)$

$\nabla L = \vec{0}$

$$\begin{cases} 2x + \lambda y = 0 \\ 2y + \lambda x = 0 \\ xy = 1 \end{cases} \quad \lambda \neq 0 \begin{cases} x = -\frac{\lambda}{2} y \\ x = -\frac{2}{\lambda} y \\ xy = 1 \end{cases} \quad \begin{cases} \lambda = 0 \\ x = y = 0 \\ 0^2 \neq 1 \end{cases}$$

$$-\frac{\lambda}{2} y + \frac{2}{\lambda} y = 0 \qquad y\left(\frac{2}{\lambda} - \frac{\lambda}{2}\right) = 0$$

$$\begin{cases} y = 0 \\ xy = 1 \end{cases} \emptyset \qquad \frac{2}{\lambda} - \frac{\lambda}{2} = 0 \qquad \begin{cases} \lambda^2 = 4 \\ x = -\frac{\lambda}{2} y \\ xy = 1 \end{cases}$$

$$\begin{cases} \lambda = 2 \\ x = -y \\ xy = 1 \end{cases} \emptyset \qquad \begin{cases} \lambda = -2 \\ x = y \\ xy = 1 \end{cases}$$

$P_1(1,1) \quad P_2(-1,-1)$

$f(P_1) = f(P_2) = 2$

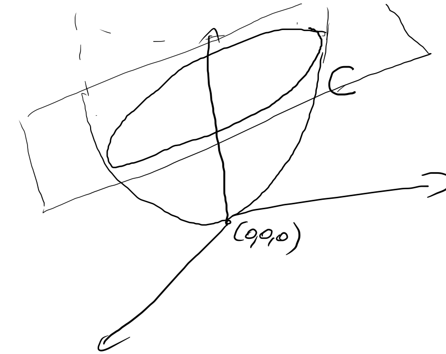
$P \in M \quad P = \left(3, \frac{1}{3}\right)$

$f(P) = 9 + \frac{1}{9} > 2$

v bodech $(1,1)$ a $(-1,-1)$ je minimum, maximum neexistuje, neboť funkce f je shora neomezená na M.

(extrémy pro dvě vazby)

Rovina $x + y + 2z = 2$ protíná paraboloid $z = x^2 + y^2$ v nějaké křivce C. Nalezněte na křivce C bod nejblíže a nejdále od počátku.



$$C = \{(x,y,z):\ g_1(x,y,z)=0,\ g_2(x,y,z)=0,\ g_1 = x+y+2z-2,\ g_2 = x^2+y^2-z\}$$

$$d(x,y,z) = \sqrt{x^2+y^2+z^2} \qquad f(x,y,z) = [d(x,y,z)]^2 = x^2+y^2+z^2$$

$$L(x,y,z) = f + \lambda_1 g_1 + \lambda_2 g_2 = x^2+y^2+z^2 + \lambda_1(x+y+2z-2) + \lambda_2(x^2+y^2-z)$$

$$\nabla L = \vec{0} \quad \begin{cases} 2x + \lambda_1 + 2\lambda_2 x = 0 \\ 2y + \lambda_1 + 2\lambda_2 y = 0 \\ 2z + 2\lambda_1 - \lambda_2 = 0 \\ x+y+2z = 2 \\ x^2+y^2 = z \end{cases} \quad \begin{cases} 2x(1+\lambda_2) + \lambda_1 = 0 \\ 2y(1+\lambda_2) + \lambda_1 = 0 \\ 2z = \lambda_2 - 2\lambda_1 \\ x+y+2z = 2 \\ x^2+y^2 = z \end{cases}$$

$\lambda_2 = -1 \Rightarrow \lambda_1 = 0$ $\quad \begin{cases} z = -\frac{1}{2} \\ x+y = 3 \\ x^2+y^2 = -\frac{1}{2} \end{cases} \quad \emptyset$

$\lambda_2 \neq -1$

$$\begin{cases} x = \dfrac{-\lambda_1}{2(\lambda_2+1)} \\ y = \dfrac{-\lambda_1}{2(\lambda_2+1)} \end{cases} \Rightarrow \begin{cases} y = x \\ x+y+2z = 2 \\ x^2+y^2 = z \end{cases} \quad \begin{aligned} &x+z = 1 \quad z = 1-x \\ &2x^2 - z = 0 \\ &2x^2 + x - 1 = 0 \\ &x = \frac{-1 \pm \sqrt{9}}{4} \ \begin{cases} -1 \\ \frac{1}{2} \end{cases} \end{aligned}$$

$P_1(-1,-1,2) \qquad P_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$

$d(P_1) = \sqrt{6} \qquad$ P_1 je bod maxim

$d(P_2) = \frac{\sqrt{3}}{2} \qquad$ P_2 je bod min.

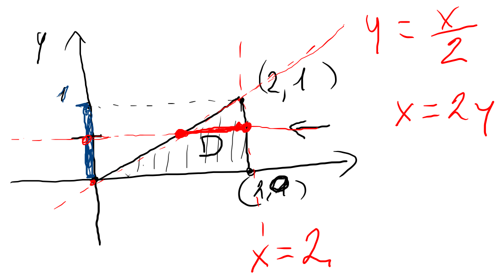
(extrémy pro dvě vazby)

Určete největší a nejmenší hodnoty funkce $f(x, y, z) = xyz$ na množině M dané podmínkami

$x + y + z = 5 \qquad$ a $\qquad xy + yz + zx = 8.$

Dvojný integrál.

1 Spočítejte $\iint\limits_D (xe^y + 2)\,dxdy$, kde D je trojúhelník s vrcholy $(0,0)$, $(2,1)$ a $(2,0)$.



$$\int_0^1 \left(\int_{2y}^2 (xe^y + 2)\,dx \right) dy =$$

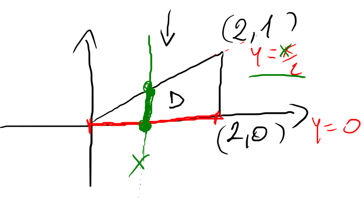
$$= \int_0^1 \left[\frac{x^2}{2} e^y + 2x \right]_{x=2y}^{x=2} dy = \int_0^1 \left(2e^y + 4 - 2y^2e^y - 4y\right) dy$$

$$= \left[2e^y + 4y - 2y^2e^y + 4ye^y - 4e^y - 4\frac{y^2}{2} \right]_0^1 = \dots$$

$$\int y^2 e^y\,dy = \begin{vmatrix} y^2 & e^y \\ 2y & e^y \end{vmatrix} = y^2e^y - \int 2ye^y\,dy = \begin{vmatrix} 2y & e^y \\ 2 & e^y \end{vmatrix}$$

$$= y^2e^y - 2ye^y + \int 2e^y\,dy =$$

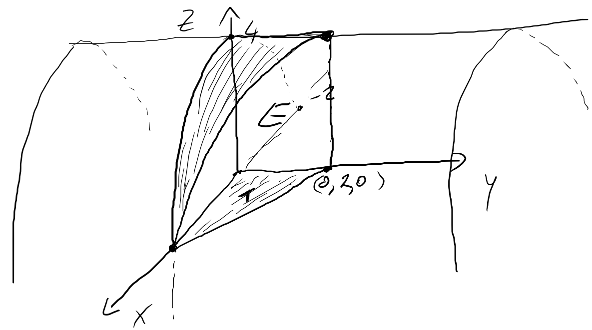
$$= y^2e^y - 2ye^y + 2e^y$$



$$\int_0^2 \left(\int_0^{x/2} (xe^y + 2)\,dy \right) dx = \dots$$

2 Určete objem tělesa pod grafem funkce $f(x,y)=4-x^2$ nad trojúhelníkem T s vrcholy $(0,0)$, $(2,0)$ a $(0,2)$.

$$\begin{cases} z=4-x^2 \\ y=k \quad k\in\mathbb{R} \end{cases} \qquad z=4-x^2$$



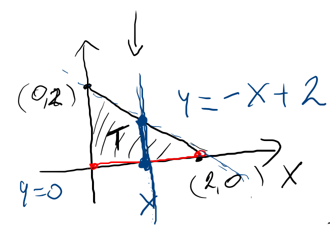
$$\text{Objem}(E)=\iint_T (4-x^2)\,dA$$

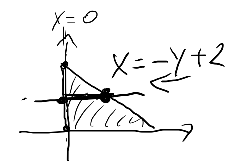
$$\int_0^2\int_0^{-x+2}(4-x^2)\,dy\,dx=$$

$$=\int_0^2(4-x^2)\big[y\big]_0^{2-x}\,dx=$$

$$\int_0^2(4-x^2)(2-x)\,dx=\int_0^2(8-4x+2x^2+x^3)\,dx=$$

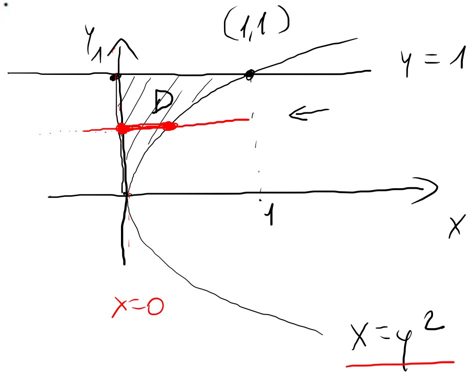
$$=\left[8x-2x^2+\frac{2x^3}{3}+\frac{x^4}{4}\right]_0^2=16-2^3+\frac{2^4}{3}+2^2.$$





$$\int_0^2\int_0^{-y+2}(4-x^2)\,dx\,dy=\dots\dots$$

Spočítejte $\iint\limits_D 3y^3e^{xy}\,dA$, kde D je omezeno křivkami $x=0$, $y=1$ a $x=y^2$.



$$\int\left(\int 3y^3 e^{xy}\,dy\right)dx$$

3 per partes !

$$\int_0^1\int_0^{y^2} 3y^3 e^{xy}\,dx\,dy = \int_0^1 3y^{\cancel{3}\,2}\left[\frac{e^{xy}}{\cancel{y}}\right]_{x=0}^{x=y^2} dy =$$

$$= \int_0^1 3y^2\left(e^{y^3}-1\right)dy = \int_0^1\left(3y^2e^{y^3}-3y^2\right)dy$$

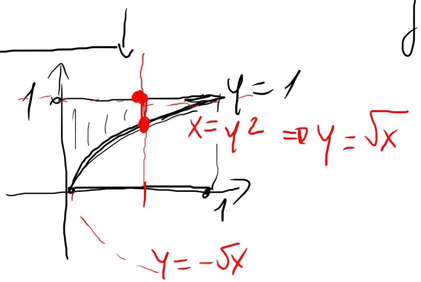
$$= \left[e^{y^3}-y^3\right]_0^1 = e-1-1 = e-2$$

$$\int e^{xy}\,dx = \left|\begin{array}{l} u = xy \\ du = y\,dx \\ dx = \frac{du}{y}\end{array}\right|$$

$$\int 3y^2e^{y^3}dy = \left|\begin{array}{l} s = y^3 \\ ds = 3y^2\,dy\end{array}\right|$$

$$\int e^s\,ds = e^s = e^{y^3}$$

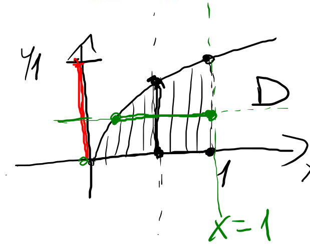
$$\int_0^1\int_{\sqrt{x}}^1 3y^3 e^{xy}\,dy\,dx$$



Změňte pořadí integrace u následujících integrálů:

$$i)\ \int_0^1 \int_0^{\sqrt{x}} f(x,y)\,dy\,dx, \qquad ii)\ \int_0^1 \int_0^{\sqrt{x-x^2}} f(x,y)\,dy\,dx,$$

$D = \{(x,y) : 0 \le x \le 1 \quad 0 \le y \le \sqrt{x}\}$



$y = \sqrt{x}$

$x = y^2$

$$i)\ \int_0^1 \int_{y^2}^1 f(x,y)\,dx\,dy$$

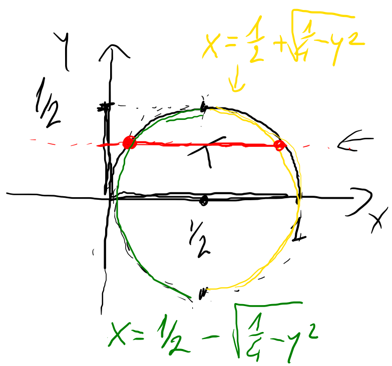
$y = \sqrt{x - x^2} \iff \begin{cases} y \ge 0 \\ y^2 = x - x^2 \end{cases}$

$x^2 - x + y^2 = 0$

$\begin{cases} \left(x - \frac{1}{2}\right)^2 + y^2 = \frac{1}{4} \\ y \ge 0 \end{cases}$

$ii)\ T = \{(x,y) : 0 \le x \le 1 \quad 0 \le y \le \sqrt{x - x^2}\}$

$$\int_0^{1/2} \int_{\frac{1}{2} - \sqrt{\frac{1}{4} - y^2}}^{\frac{1}{2} + \sqrt{\frac{1}{4} - y^2}} f(x,y)\,dx\,dy$$

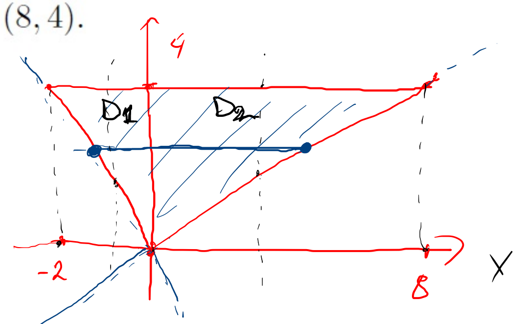


$(x - 1/2)^2 = 1/4 - y^2$

$x - \frac{1}{2} = \pm\sqrt{\frac{1}{4} - y^2} \qquad x = \frac{1}{2} \pm \sqrt{\frac{1}{4} - y^2}$

Spočítejte $\iint_D e^{y^2+1}\, dxdy$, kde D je trojúhelník s vrcholy $(0,0)$, $(-2,4)$ a $(8,4)$.

$$\int_0^4 \int_{-y/2}^{2y} e^{y^2+1}\, dx\, dy$$



$y = -2x$

$y = x/2$

$x = -\frac{y}{2}$

$D = D_1 \cup D_2$

$x = 2y$

$$\iint_D (\quad)\, \underline{\underline{dy\,dx}} = \iint_{D_1} (\quad)\, dy\,dx + \iint_{D_2} (\quad)\, dy\,dx$$

1. Zaměňte pořadí integrace v následujících dvojnásobných integrálech:

   (a) $\int_0^4 \int_{3x^2}^{12x} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$;

   (b) $\int_{-1}^2 \int_0^{e^{-x}} f(x,y)\,\mathrm{d}y\,\mathrm{d}x$.

2. Vypočtěte $\int_M f(x,y)\,\mathrm{d}A$, jestliže

   (a) $f(x,y) = x\cos y$ a M je množina ohraničená křivkami $y = 0$, $y = x^2$, $x = 1$;

   (b) $f(x,y) = \frac{y}{x^2+y^2}$ a M je množina ohraničená křivkami $y = x$, $y = 2x$, $x = 1$, $x = 2$;

   (c) $f(x,y) = |y - \sin x|$, $M = [0,\pi] \times [0,1]$.

3. Nalezněte obsah plochy ohraničené křivkami $y = -3x + 6$, $y = 0$ a $y = 4x - x^2$, kde $x \in [0,2]$.

4.